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Vertex-oblique graphs

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Abstract

Let x be a vertex of a simple graph G. The *vertex-type* of x is the lexicographically ordered degree sequence of its neighbors. We call the graph G *vertex-oblique* if there are no two vertices in $V(G)$ which are of the same vertex-type. We will show that the set of vertex-oblique graphs of arbitrary connectivity is infinite.

Keywords: Oblique graphs; Asymmetric graphs

1. Introduction

We use [1] for terminology and notation. Let $G = (V, E)$ be a simple graph. By $d(x)$ we denote the degree of the vertex x. It is a well-known fact that in each simple graph, there exists a vertex degree, that occurs at least twice. We will define a type of a vertex by taking into consideration the vertex degrees of the neighbors of each vertex, too. The *vertex-type* $(d_1, \ldots, d_{d(x)})$ of a vertex $x \in V(G)$ is the lexicographically ordered degree sequence of the vertices adjacent to x, where $d_i \leqslant d_j$ if $i \leqslant j$. Now one can ask the question, whether there must still be two vertices of the same type. It is easy to see that this need not be the case. But on the other hand, a Ramsey principle states that in every large enough structure, some regularity can be found. So may be the set of graphs, where each type occurs at most once is finite.

A graph is called *vertex oblique* if there are no two vertices of the same type in $V(G)$. If one compares the vertex types of two vertices x_1, x_2 it is convenient to introduce an ordering $\prec$ of the vertices. The vertex x_1 is said to be of lower order than x_2, $x_1 \prec x_2$, if either it has a lower degree, or, if the degrees are equal, the type vector of x_1 is lexicographically smaller than the type vector of x_2. With the definition of the ordering $\prec$ it is possible to formulate the graph-property of being vertex oblique in another way: a graph G is vertex oblique if and only if there is a total ordering $\prec$ on its vertex-set.

2. Some first observations

Lemma 1. *There is no vertex oblique tree on more than one vertex.*

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Proof. Consider a tree T with $|V(T)|>1$ that is vertex oblique. Then remove all end vertices of the tree. Since they are all of degree one, there cannot be two of them adjacent to the same vertex, and they cannot be adjacent to each other. Therefore after deletion of the $l \geqslant 2$ end vertices a tree T' on at least l vertices remains. Every end vertex of T' was adjacent to an end vertex of T in T, otherwise it would have been deleted. That means, it was of degree 2 in T. In this case, T' can only have one end vertex, otherwise there must have been more than 1 vertex of type (2) in T. But since T' is a tree on more than one vertex, it has more than one end vertex. □

Lemma 2. *If G is vertex oblique, then its complement $\overline{G}$ is vertex oblique, too.*

Proof. We will prove the claim by contradiction. Assume the graph G on n vertices is vertex oblique and $\overline{G}$ is not. Then there must be two vertices x and y in $V(\overline{G})=V(G)$ which are of the same type in $\overline{G}$. Two vertices are of the same type if and only if the multisets consisting of the degrees of their neighbors are equal. Whether the vertices x and y are connected does not matter, because if they are connected, in both type vectors occurs a number $d(x)$ corresponding to x or y, or no number corresponding to any of the two vertices occurs in either of the type vectors.

Let us distribute the remaining vertices into four sets:

$A := N_{\overline{G}}(x)\backslash(N_{\overline{G}}(y)\cup\{y\})$-private neighbors of x,
$B := N_{\overline{G}}(y)\backslash(N_{\overline{G}}(x)\cup\{x\})$-private neighbors of y,
$C := N_{\overline{G}}(x)\cap N_{\overline{G}}(y)\}$-common neighbors of x and y,
$D := V(\overline{G})\backslash(N_{\overline{G}}(x)\cup N_{\overline{G}}(y)\cup\{x, y\})\}$-common non-neighbors of x and y,

$N_{\overline{G}}(x)\backslash\{y\}=A\cup C$,
$N_{\overline{G}}(y)\backslash\{x\}=B\cup C$.

Since x and y are of the same type in $\overline{G}$, the multisets of vertex degrees of A and B must be equal.

In G the neighborhoods of x and y are the following:

$N_G(x)\backslash\{y\}=B\cup D$,
$N_G(y)\backslash\{x\}=A\cup D$.

The multisets of vertex-degrees of A and B are still the same, because every vertex-degree $d_{\overline{G}}(v)$ in the graph G is replaced by $d_G(v)=n-1-d_{\overline{G}}(v)$ in the graph G. Thus the degree-multisets for x and y are equal, and therefore their types are equal too. But this is a contradiction to the assumption that G is vertex oblique. □

The smallest examples of vertex-oblique graphs apart from the trivial case K_1 are G_0 and G_1 (see Section 3) and their complements. These are the only such graphs on six vertices. Given a vertex-oblique graph G of a certain connectivity k which has some further properties, it is easy to construct another such graph by applying the following construction.

Construction 1. *Start*: Given a graph G with the following properties:

(1) G and $\overline{G}$ are vertex oblique.
(2) If $x_1 \prec x_2 \prec \cdots \prec x_k$ are the k vertices of highest order in G, then $d_G(x_1)<d_G(x_2)<\cdots<d_G(x_k)$ and $\forall w \in V(G)\backslash X\colon d_G(w)<d_G(x_1)$, where $X=\{x_1,\ldots,x_k\}$
(3) G is k-connected.
(4) The maximum vertex degree $\Delta(G)\leqslant|V(G)|-2$.
(5) In case $k=1$, the vertex of maximum degree has no neighbor of degree 1.

Construction: do the following:
Step 1: Add a K_k with vertices $y_1,\ldots,y_k$.
Step 2: Remove a maximum matching from the K_k. If k is odd, remove no edge incident to y_k
Step 3: Insert the edges (x_i, y_i) for $i=1,\ldots,k$.
Step 4: Add a vertex z connected with all vertices of G apart from $x_1,\ldots,x_k$ and with $y_1,\ldots,y_k$.

Theorem 3. *The resulting graph H is a graph on $|V(G)|+k+1$ vertices. It still has the properties of G. In addition to this, the graph $\overline{H}$ is k-connected.*

Proof. To prove the properties we distribute the vertex set of H into four parts: $V(H)=W\cup X\cup Y\cup\{z\}$, where $X=\{x_1,\ldots,x_k\}$, $Y=\{y_1,\ldots,y_k\}$ and $W=V(G)\backslash X$.

Since G was k-connected, $k \leqslant d_G(v) \leqslant |V(G)| - 2$ for all $v \in V(G)$. By the construction the degree of every vertex of G increases by one. Still $k+1 \leqslant d_H(v) \leqslant |V(G)| - 1$ for all $v \in V(G)$. If k is even each vertex of Y has in H a degree of k. If k is odd each vertex of $Y \backslash \{y_k\}$ has in H a degree k and $d_H(y_k) = k+1$. Obviously the following is true:

$d_H(x_1) < d_H(x_2) < \cdots < d_H(x_k) < |V(G)| = d_H(z)$. Every vertex of W still has a smaller degree than x_1, and every vertex of Y has a degree k or $k+1$, which is also smaller than the degree of x_1. Therefore H has the property (2). The graph H has also property (5) since the vertex z of maximum degree is not adjacent to any vertex of X.

The vertices $x_1 \prec x_2 \prec \cdots \prec x_k \prec z$ are the vertices of highest degree (and order) in H. All other vertices have a lower order and therefore a different type.

Each vertex of W gets one further neighbor z by the construction. Their degree rises by one and in each type vector every number is increased by one and it is prolonged by a further number n. Thus, all these vertices keep their ordering and are therefore still of different types.

If $k \geqslant 3$ is odd $d_H(y_k) = k+1$ and has a neighbor of degree k (e.g. y_1) which no other vertex of degree $k+1$ can have. If $k=1$ the type of y_1 remains unique if x_1 has no neighbor of degree one in G, which is ensured by property (5). The new graph H still has property (5), since it contains no vertex of degree 1.

All other vertices in Y have degree k and are therefore of lower order than any vertex of W. They also have all different types, because every vertex of Y is adjacent to exactly one vertex of X with a unique degree.

Hence, H is vertex oblique. Because of Lemma 2, $\overline{H}$ is vertex oblique, too.

It remains to show that H and $\overline{H}$ are k-connected. We will show this by proving the existence of k vertex-disjoint paths between any two vertices in H, $\overline{H}$ respectively.

- $u, v \in W$
  Graph H:
  Since G is k-connected, there are k paths between u and v in G, and therefore in H, too.
  Graph $\overline{H}$:
  k paths (u, y_i, v) for $i = 1, \ldots, k$.
- $u \in W, v = x_i$
  Graph H:
  Since G is k-connected, there are k paths between u and v in G, and therefore in H, too.
  Graph $\overline{H}$:
  $k-1$ paths (u, y_j, x_i) for $i \neq j$, and one path (u, y_i, x_j, z, x_i) for an arbitrary $j \neq i$. This works only if $k > 1$.
  If $k=1$ there is, because of condition (4), at least one vertex w in $V(G)$ which is not adjacent to x_1. If $u = w$ then we have the path (u, x_1) otherwise (u, y_1, w, x_1).
- $u \in W, v = y_i$
  Graph H:
  y_i is adjacent to all but one vertex of Y. Therefore there exist $k-1$ paths of length at most 2 from y_i to $k-1$ different vertices of X. Because G is k-connected, there are $k-1$ vertex-disjoint paths from u to these vertices in G. The last path is (u, z, y_i).
  Graph $\overline{H}$:
  In W there are at least k different vertices $u_1, \ldots, u_k$ apart from u, therefore $k-1$ paths (u, y_j, u_j, y_i) for $i \neq j$ and the kth path is (u, y_i).
- $u \in W, v = z$
  Graph H:
  Since G is k-connected, there are k vertex-disjoint paths from u to $x_1, \ldots, x_k$ which can be prolonged via $y_1, \ldots, y_k$ to $u - z$ paths.
  Graph $\overline{H}$:
  k paths (u, y_i, x_j, z), where $i \neq j$. This only works if $k > 1$.If $k=1$ there is one vertex w, which is not adjacent to x_1 in G. If $u = w$ we have the path (u, x_1, z) otherwise (u, y_1, w, x_1, z).
- $u = x_i, v = x_j, i \neq j$ (only relevant for $k \geqslant 2$)
  Graph H:
  Since G is k-connected, there are k vertex-disjoint paths between x_i and x_j in G and therefore in H, too.
  Graph $\overline{H}$:
  $k-2$ paths (x_i, y_l, x_j), where $l \neq i, j$.

One path (x_i, y_j, w, y_i, x_j), where w is an arbitrary vertex of W.
One path (x_i, z, x_j).

- $u = x_i$, $v = y_j$
Graph H:
There are $k-1$ paths of length at most 2 from y_j to $k-1$ different vertices of X and since G is k-connected there are $k-1$ vertex-disjoint paths from these vertices to x_i in G. Note that one of these paths can be the single vertex x_i itself.
In this way one gets $k-1$ $x_i - y_j$ paths. By taking the shortest possible such paths one can ensure that at most $k-1$ neighbors of x_i are contained in the paths. But as x_i is of degree at least k in G there is still one further neighbor w of x_i which is not contained in any of the paths. If $w \in W$ then there is a path (x_i, w, z, y_j). If $w = x_l$ for some l then it is the vertex of X which is not connected by a path of length at most 2 with y_j. Therefore y_l is also not contained in any of the paths. That means that there is a new path (x_i, x_l, y_l, z, y_j), which is internally disjoint from each of the other paths.
Graph $\overline{H}$:
Let us suppose $i = j$. Because of property (4) there must be one vertex t in $V(G)$ which is not adjacent to x_i in G or H. It is also not adjacent to y_i in H. Therefore one path in $\overline{H}$ is (x_i, t, y_i)
There are $k-1$ further paths (x_i, y_l, w_l, y_j), where $l = 1, \ldots, k$, $l \neq i$ and $w_1, \ldots, w_k$ are different vertices of W distinct from t.
If $i \neq j$ there is one path (x_i, y_j) and $k-2$ paths (x_i, y_l, w_l, y_j), where $l = 1, \ldots, k$, $l \neq i, j$ and $w_1, \ldots, w_k$ are different vertices of W. The last path is (x_i, z, x_l, y_j), where $l \neq i, j$. This works only if $k > 2$. If $k = 1$ this case cannot occur and if $k = 2$ the remaining path is (x_i, z, x_j, y_i, y_j).
- $u = x_i$, $v = z$
Graph H:
One path (x_i, y_i, z).
$k-1$ paths (z, y_j, x_j) between z and the other $k-1$ vertices of X. Because G is k-connected, there must be $k-1$ internally disjoint paths in G from these $k-1$ vertices of X to x_i. Together they form $k-1$ $z - x_i$ paths.
Graph $\overline{H}$:
One path (x_i, z).
The subgraph H' of $\overline{H}$ induced on the vertices $(X \backslash \{x_i\}) \cup (Y \backslash \{y_i\})$ contains a complete $K_{k-1,k-1}$ reduced by a perfect matching. So if $k \geqslant 3$, another perfect matching remains. For every matching edge (x_l, y_m) one gets a corresponding path (x_i, y_m, x_l, z) in $\overline{H}$. In this way the remaining $k-1$ paths are conceived. If $k = 1$ we already have the path. If $k = 2$ there is one vertex w of $V(G)$ which is not adjacent to x_i in G. If it is the other vertex x_j of X we have the path (x_i, x_j, z) otherwise (x_i, w, y_i, x_j, z).
- $u = y_i$, $v = y_j$, $i \neq j$ (only relevant for $k \geqslant 2$)
Graph H:
One path (y_i, z, y_j).
If $k = 2$ one further path (y_1, x_1, P, x_2, y_2), where P is an $x_1 - x_2$ path in G.
If $k = 3$ we take the same second path and there is one path of length at most 2 that connects y_i and y_j and consists only of vertices of Y.
If $k \geqslant 4$ then:
if y_i and y_j are not adjacent, then $k-2$ paths (y_i, y_l, y_j) for $l = 1, \ldots, k$, $l \neq i, j$ and one path (y_i, x_i, P, x_j, y_j), where P is an $x_i - x_j$ path in G.
If y_i and y_j are adjacent they have at least $k-4$ common neighbors in Y which form $k-4$ internally disjoint $y_i - y_j$ paths. One further path is (y_i, y_j). There is still one neighbor y_l of y_i which is not contained in any of the $k-4$ paths and one neighbor y_m of y_j which is also not contained. If one of them is a common neighbor there is a path via this vertex otherwise y_l and y_m are adjacent and we have the path (y_i, y_l, y_m, y_j). The last path is (y_i, x_i, P, x_j, y_j), where P is an $x_i - x_j$ path in G.
Graph $\overline{H}$:
k paths (y_i, w_l, y_j), where $w_1, \ldots, w_k$ are different vertices of W.
- $u = y_i$, $v = z$
Graph H:
One path (y_i, z).

At least $k-2$ paths (y_i, y_j, z).
One path (y_i, x_i, P, w, z), where P is an arbitrary $(x_i - w)$ path in G and w is an arbitrary vertex of W.
Graph $\overline{H}$:
$k-1$ paths (y_i, x_j, z) for $i \neq j$.
One path (y_i, w, y_j, x_i, z) where $j \neq i$ and w is an arbitrary vertex of W.
This construction works for $k \geqslant 2$. If $k = 1$ then w is the vertex not adjacent to x_1 in G and we have the path (y_1, w, x_1, z) in $\overline{G}$.

Thus, H and $\overline{H}$ are k-connected. This completes the proof of Theorem 3. □

Since there are graphs, e.g. G_1, that fulfill the conditions for the application of construction 1, it is possible to answer the initial question.

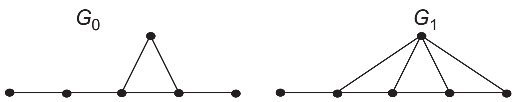
Corollary 4. *There are infinitely many vertex-oblique graphs.*

3. Super-vertex-oblique decomposition of K_n into two parts

As one can see, construction 1 not only delivers a sequence of vertex-oblique graphs, it also preserves the connectivity. A trivial observation is that every vertex-oblique graph G on n vertices induces a decomposition of the complete graph K_n on n vertices into two parts, namely G and its complement $\overline{G}$. We call such a decomposition *super-vertex-oblique* if not only each part is vertex-oblique, but they have no vertex-type in common, too. For large enough n such a decomposition of K_n exists even in such a way that each part is *c-connected*, which will be shown in the next theorem.

Theorem 5. *For every c there is a bound n_c, such that for each $n \geqslant n_c$ there exists a super-vertex-oblique decomposition of the complete graph K_n into two c-connected parts.*

Proof.



The graph G_1 fulfills all requirements to apply construction 1 for $k=1$. Consider the graph sequence $\{G_i | i = 1, 2, \ldots\}$, where G_i is derived from G_{i-1} by applying construction 1 for $k = 1$.

In every graph, the degree of each vertex increases by 1 and an additional vertex of unique maximum degree is added. Therefore, since in G_1 the highest degree occurred only once, in G_i the highest i vertex degrees occur only once.

Construction 2. Let the desired connectivity c be greater than 3. This is no restriction since a c-connected graph is $c-1$-connected, too.

Step 1: Consider the graph $H_1 = G_c$. Its highest c vertex degrees occur only once. $|V(H_1)| = 2c + 4$.

Step 2: Let H_2 be the disjoint union of $G_1,\ldots,G_l$.
Choose l in such a way that $l^2 + 3l - 6 > 2c$.
Then H_2 has $\sum_{i=1}^{l}(2i+4) = l^2 + 5l$ vertices and a maximum degree $\Delta(G_2) = 2 + 2l$.
H_2 is a (disconnected) vertex-oblique graph, because:
Assume there are two vertices x and y of the same type. Then they must be in different components, since each component is a vertex-oblique graph.

Let x belong to the graph G_i and y to G_j, where $i < j$. Then by construction y is either of degree $1 + 2j$ or $2 + 2j$ or it has a neighbor of one of these two degrees. On the other hand x belongs to G_i, where the maximum degree is $2 + 2i$ which is smaller then $1 + 2j$. Thus the vertices x and y cannot be of the same type.

Step 3: Construct H by taking the graphs H_1 and H_2 and connect each vertex of H_1 with every vertex of H_2.

The resulting graph H is at least $2c + 4$-connected, since it contains the complete bipartite graph K_{2c+4,l^2+5l} as a subgraph, that already contains all vertices of H. The vertices of H_1 are of different types in H, since any two vertices of H_1 have the same new neighbors (namely all vertices of H_2). The same is true for H_2. In H the maximum degree of a vertex in H_2 is $\Delta(H_2) + |V(H_1)| = 2c + 2l + 6$, where $\Delta(H_2)$ is the maximum degree in H_2. The minimum degree of a vertex of H_1 in H is greater than $|V(H_2)| = l^2 + 5l$ and because of the choosing of l this number is greater than the maximum degree of a vertex belonging to H_2. Therefore a vertex of H_1 cannot be of the same type as a vertex of H_2. Thus H is vertex oblique. Moreover the vertices with the highest c degrees are the same as they were in H_1 and each of the highest c vertex degrees in H occurs only once.

In this way one gets a graph which fulfills the conditions needed to apply construction 1 for $k = c$. Applying the construction once, one gets a graph H^n on n vertices, which has a c-connected complement. Starting from this graph one gets a sequence of vertex-oblique c-connected graphs, where the complements are c-connected too, on $m = n + p(c+1)$ vertices for $p \geqslant 1$. That means, between two consecutive graphs of the sequence, there is a gap of c vertices. We will fill this gap by a different construction.

Let H^m be a graph of the sequence on $m = n + p(c + 1)$ vertices. Consider the $c + 1$ vertices z and $x_1, \ldots, x_c$ from construction 1. Then add successively vertices of degree c connected with the vertices:

$z, x_c, \ldots, x_2,$

$z, x_c, \ldots, x_3, x_1,$

$\vdots$

$z, x_{c-1}, \ldots, x_1.$

In this way one can get the c graphs $H^{m+1}, \ldots, H^{m+c}$, with numbers of vertices from $m + 1$ up to $m + c$.

The new vertices are, together with the vertices $y_1, \ldots, y_c$, the only ones of degree c. Moreover they are all of different types, since they have no neighbor of degree c like the y_i vertices and they are all connected with different c-tuples of vertices of different degrees. It is easy to see that in each step the inequality $d(x_1) < \cdots < d(x_c) < d(z)$ remains true. Therefore they have always different types. All other vertices keep their ordering. In each step a vertex of degree c is added. Therefore the graph remains c-connected. In the complement a vertex of degree greater than c is added, so it remains c-connected, too.

It remains to show that the decompositions in the constructed sequences are super-vertex-oblique. To prove the existence of the decompositions it is sufficient to restrict to even c. Let H^{m+p} be a graph on $m + p$ vertices of the sequence for such a connectivity c, where $p \in \{0, \ldots, c\}$. In this case construction 1 has been applied at least once. The sets X, Y and the vertex z shall be the corresponding sets from the last application of construction 1. Assume there are vertices u and v, where the type of u in H^{m+p} is the same as the type of v in $\overline{H}^{m+p}$.

Case 1: $v \notin \{z\} \cup Y$

Then v is adjacent to at least 2 of the vertices y_1, y_2, y_3 which are of degree $m + p - c - 1$ in $\overline{H}^{m+p}$. But in H^{m+p} there is only one vertex of such a high degree, namely z.

Case 2: $v \in Y$

Then $d_{\overline{H}^{m+p}}(v) = m + p - c - 1$ and since c is even v is adjacent to another vertex of Y in $\overline{H}^{m+p}$ which is also of degree $m + p - c - 1$. But in H^{m+p} there is only one vertex of this degree.

Case 3: $v = z$

Then $d_{\overline{H}^{m+p}}(z) = c$. The neighbors of z in $\overline{H}^{m+p}$ are $x_1, \ldots, x_c$, none of which is of degree $m + p - c - 1$. But all vertices of degree c in H^{m+p}, namely those of Y and the vertices from the filling construction, are adjacent to z in H^{m+p}, which is of degree $m + p - c - 1$ in H^{m+p}. □

Remark. Using the described algorithm one gets a bound n_c of order $5c + o(c)$. For small values of c one can apply construction 1 directly on some small initial decompositions. In this way one can improve the bounds to $n_1 = 6$, $n_2 = 8$, $n_3 = 11$ and $n_4 = 15$.

4. Open problems

- In [3–5] it has been shown that for other types of obliqueness the set of oblique polyhedral graphs is finite. It remains to investigate whether the same is true in the vertex-oblique case.
- Should there only be finite many polyhedral vertex-oblique graphs, are there at least infinitely many vertex-oblique graphs with bounded average degree?

- We conjecture that for every d there is a super-vertex-oblique decomposition of K_n into d connected parts, if n is large enough. We already have a construction for $d \leqslant 7$. But it uses computers to find some initial graphs.
- We conjecture that if there is a super-vertex-oblique decomposition of K_n into $d \geqslant 3$ parts, there is one into $d - 1$ parts, too.
- We conjecture that if a super-vertex-oblique decomposition of K_n into d parts exists, such a decomposition can be found for K_{n+1}, too.

One could also ask whether there are self-complementary vertex-oblique graphs. This question has been solved recently by Alastair Farrugia [2], who could show that this is not the case. But he constructed a sequence of vertex-oblique graphs, where each graph has the same set of vertex-types as its complement.

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